Periodic solitons in optics

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Periodic (higher-order) solitons are presumed to be a by-product of integrability and thus, in optics, applicable only to one-dimensional beams, propagating in a Kerr (cubic) nonlinear medium. Here, their existence is demonstrated numerically both in one and two transverse dimensions, propagating in a medium that differs radically from a Kerr nonlinearity. Our motivation comes from linear physics. This also provides physical understanding of higher-order solitons.

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Chiao, Garmire, and Townes showed the existence of self-guided optical beams (fundamental spatial solitons) [1]. These are stationary solutions of the wave equation and are characterized, in general, by fields

$$E(x,y,z)=\Psi(x,y)\exp(i\beta z). \quad (1)$$

Here, the (real) transverse field distribution Ψ and the associated propagation constant β are related via the expression $\left\{\nabla_t^2+k^2n^2(\Psi^2)-\beta^2\right\}\Psi=0$, where ∇_t^2 denotes the transverse Laplacian, $k=2\pi/\lambda$, λ is the free-space wavelength, and $n^2(\Psi^2)$ specifies the dependence of the refractive index on intensity.

One monumental achievement of the inverse scattering technique is to demonstrate the existence of second-order solitons, i.e., self-guided beams whose intensity profile changes periodically in the direction of propagation [2–4]. However, the inverse scattering technique applies only to integrable systems which, in optics, is restricted to the nonlinear Schrödinger equation or, equivalently, to one-dimensional beams propagating in a Kerr (cubic) nonlinearity.

In light of the above remarks, we demonstrate an interesting fact. Optical periodic spatial solitons, of both one and two transverse dimensions, propagate in the threshold nonlinearity — a nonlinearity which differs radically from the Kerr nonlinearity. There is some radiation, but it is insignificant over a period. Indeed, after numerically beam propagating these periodic solitons for one hundred periods we typically observe only about 1% radiation loss. We emphasize that our aim in this paper is to demonstrate the first numerical evidence of periodic solitons in a non-Kerr material and not to provide a general prescription for generating such solitons.

The motivation for this work comes from the need to explain the salient physics underlying the recent experimental observations of periodic (higher-order) solitons [5–7]. Our initial reasoning and physical intuition comes directly from the perspective of *linear* physics. Thus, quite apart from our new results, we also introduce a conceptual approach, which provides physical understanding of periodic solitons. Indeed, prior to implementing the linear perspective discussed here, we found the subject of higher-order solitons mystifying. Why, for example, does scaling up [3,4] a fundamental ($N=1$) soliton eventually result in it becoming a bound periodic soliton?

To initiate our approach to periodic spatial solitons, we first give a concise overview of the linear perspective to nonlinear wave optics. The physics of self-guided beams is particularly transparent from this perspective [1,8]: stationary solutions of the wave equation are simply the modes of the soliton-induced (linear) optical waveguide that is characterized by $n^2(|E|^2)$. The optical waveguide is uniform in the direction of propagation z but it has a graded refractive index profile $n(x,y)$ transverse to the direction of propagation [9].

More importantly, the linear perspective enables conceptual leaps. For example, all self-guided optical beams that have been investigated since the pioneering work of Chiao *et al.* [1] have been stationary solutions. Nevertheless, if a self-guided beam is a mode of the linear optical waveguide it induces, why can it not also be two orthogonally polarized modes a and b of the induced *multimoded* waveguide? This generalized self-guided beam has a field of the form

$$\boldsymbol{E}(x,y,z)=\Psi_a(x,y)\exp(i\beta_a z)\hat{\boldsymbol{e}}_a+\Psi_b(x,y)\exp(i\beta_b z)\hat{\boldsymbol{e}}_b, \quad (2)$$

where $\{\nabla_t^2+k^2n^2(|\boldsymbol{E}|^2)-\beta_a^2\}\Psi_a=0$ and $\{\nabla_t^2+k^2n^2(|\boldsymbol{E}|^2)-\beta_b^2\}\Psi_b=0$. The orthogonally polarized modes ($\hat{\boldsymbol{e}}_a\cdot\hat{\boldsymbol{e}}_b^*=0$) ensure that the soliton intensity remains uniform in the direction of propagation but the polarization state now changes as the soliton evolves. These *dynamic solitons* do exist and, in fact, have an exact analytical description for both the Kerr and threshold nonlinearities [10].

Armed with this linear perspective, we can now anticipate the existence of periodic solitons. Again, suppose that a beam propagating through a nonlinear medium is composed of two modes of the multimoded linear waveguide it induces, but now let the two modes be identically polarized so that $\hat{\boldsymbol{e}}_a\cdot\hat{\boldsymbol{e}}_b^*=1$ in Eq. (2). Because of modal beating, the intensity of the beam would change sinusoidally in the direction of propagation. This elementary picture of periodic solitons provides physical insight. For example, it suggests that higher-order solitons result only if the induced waveguide is multimoded. This is consistent with the fact that periodic solitons occur only when the fundamental ($N = 1$) soliton has been sufficiently scaled up [3,4]. It is also consistent with the axially varying intensity profiles of higher-order solitons, e.g., the second-order soliton appears qualitatively like the beating of the first two even modes.

We now give an example of this linear perspective for predicting periodic spatial solitons in non-Kerr material, both in one and two transverse dimensions. To do this, we consider the threshold nonlinearity which differs radically from the Kerr nonlinearity. It supports bistable self-guided beams (spatial solitons) in both one [11] and two [8] transverse dimensions and is characterized by $n^2(|\boldsymbol{E}|^2) = n_\infty$ for $|\boldsymbol{E}|^2 < I_{th}$ and $n^2(|\boldsymbol{E}|^2) = n_0$ for $|\boldsymbol{E}|^2 > I_{th}$, where $n_0 > n_\infty$. Spatial solitons of the threshold nonlinearity induce step-index profile (linear) optical waveguides whose modes are well known [12].

We first consider one-dimensional beams. A beam characterized by the field $\Psi(x)$ is a fundamental ($N = 1$) spatial soliton when it is a mode of the step-profile (slab) waveguide it induces. Now, consider a beam composed of two like-polarized modes a and b of a slab waveguide such that

$$E(x,z) = \Psi_a(x)\exp(i\beta_a z) + \Psi_b(x)\exp(i\beta_b z), \qquad (3)$$

where the power $P_a = \int_{A_\infty} \Psi_a^2 \; dA$ of mode a and $P_b = \int_{A_\infty} \Psi_b^2 \; dA$ of mode b are independent and arbitrary. This composite field is a candidate for a periodic soliton provided P_b/P_a is sufficiently small for the field to induce a single step-profile waveguide. Furthermore, if the periodic soliton is to have transverse even symmetry, then Ψ_a and Ψ_b must be even (symmetric) modes.

To give a specific example, we use numerical beam propagation to determine what happens when a beam, initially specified by $E(x, z = 0) = \Psi_a(x) + \Psi_b(x)$, is launched in a nonlinear threshold medium. Here, Ψ_a and Ψ_b denote the first and second even modes of a step-profile slab waveguide, respectively, while the relative power in each of these modes is taken to be $P_b/P_a = 0.0848$. The (initial) induced waveguide is characterized by the parameter $V = 4$, where $V = k\rho\sqrt{n_0^2 - n_\infty^2}$, ρ is the waveguide half-width and n_0, n_∞ represent the maximum and minimum refractive indices of the threshold nonlinearity, respectively. According to the physical argument given above, this beam propagates with a periodic intensity profile. Furthermore, in the threshold nonlinearity considered here, a beam with a periodic intensity induces a step-profile waveguide whose width changes periodically in the direction of propagation. Indeed, Fig. 1(a) demonstrates that the induced waveguide appears periodic, as does the intensity profile of the beam, Fig. 1(b). This beam is clearly stable to propagation; other numerical simulations with different launch conditions (i.e., different P_b/P_a values) were found to yield unstable beams that did not propagate.

Entirely analogous results were found for beams of circular cross section. For example, by launching the first two circularly symmetric modes of a $V = 5$ step-profile circular waveguide such that $P_b/P_a = 0.1344$, the results are qualitatively the same as those presented in Fig. 1. The notation here is identical to the example given above but ρ now denotes the waveguide radius.

In summary, we have demonstrated numerically that spatial solitons of second-order can propagate in both one and two transverse dimensions in a medium which differs dramatically from the Kerr nonlinearity. Furthermore, these results are fully anticipated from the elementary physics of modal beating in a linear optical waveguide. Consequently, to have a second-order soliton, the initial beam must induce a waveguide which propagates the first and second even modes only. Consistent with this fact, we have also shown that second-order threshold solitons can result from scaling up fundamental ($N = 1$) solitons. This follows because the induced waveguide becomes multimoded for a sufficiently large amount of scaling. Indeed, these second-order solitons are generated by a variety of initial conditions, provided the resulting in-

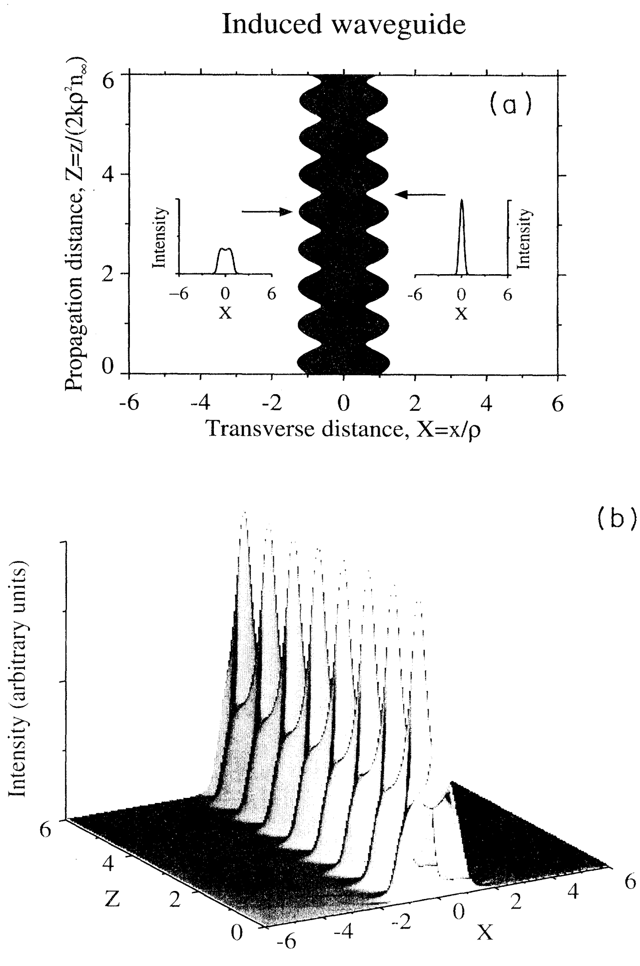

FIG. 1. One-dimensional periodic soliton propagating in a threshold nonlinearity. The first two even modes of a $V = 4$ step-profile planar waveguide are launched with $P_b/P_a = 0.0848$ to induce a self-consistent waveguide initially. (a) Induced periodic waveguide with $n(x) = n_0$ over the shaded region and $n(x) = n_\infty$ elsewhere. The two insets illustrate the intensity profiles of the periodic soliton at the maximum and minimum transverse dimensions of the induced waveguide. (b) Intensity profile of the periodic soliton as it evolves.

duced waveguide becomes multimoded, e.g., by colliding two $N = 1$ solitons [13].

We have shown numerically that second-order solitons propagate over many periods in a threshold nonlinear medium, but do they propagate indefinitely without changing their form? To answer this question theoretically, we again appeal to the linear equivalence principle which, in the present context, is stated as follows: a periodic soliton must be a Floquet or periodic mode of the periodic (linear) optical waveguide it induces [14]. This elementary, yet exact, concept fully describes periodic solitons. Consequently, we can borrow results freely from the literature of linear periodic waveguides [15], from which we learn that periodic modes are, in general, leaky for structures with spatial wavelength perturbations relevant to our present study. Furthermore, first-order perturbation theory shows that the radiation comes from the second even mode [16].

To test this prediction, we have numerically beam propagated the soliton discussed in Fig. 1 for one hundred periods which revealed a total radiation loss of about 1%. The consequences of this small amount of radiation are demonstrated in Fig. 2. Consistent with perturbation theory, the power in the second even mode couples to the fundamental (first even) mode. This, in turn, causes the amplitude of the periodic perturbation to decrease. It is not possible for the fundamental-mode power to couple back into the second even mode because this would violate the fact that the fundamental mode is stable (at $V = 4$) to arbitrary perturbations [11]. Thus, second-order spatial solitons are leaky when propagating in a threshold nonlinearity but the leakage is insignificant over distances that are experimentally meaningful [18]. Our argument from linear physics suggests that all periodic modes must radiate in the spatial wavelength region of interest. Yet, this would appear to contradict the well-known result of inverse scattering: second-order one-dimensional solitons do not radiate in a Kerr nonlinearity. Is it possible to contrive a shape of periodic distortion such that a linear optical waveguide does not have radiating periodic modes? Apparently, yes, and this shape is the one induced by a second-order soliton in a Kerr nonlinearity [19].

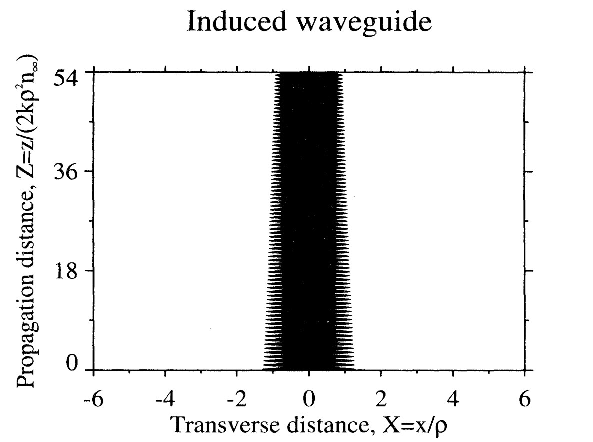

FIG. 2. As in 1(a) but showing the gradual power transfer from the second to the first even mode after a much longer propagation distance.

In conclusion, we have shown that periodic (second-order) optical solitons of one and two transverse dimensions can propagate in both a Kerr nonlinearity and a material differing radically from the Kerr nonlinearity. Although they are leaky, the radiation is negligible per period and, in particular, is insignificant over distances considered experimentally meaningful [18]. These conclusions follow from the elementary physics of linear waves. More importantly, the linear perspective also explains the physics of higher-order solitons by showing that the nonlinear-induced waveguide must be multimoded to support higher-order solitons. On the other hand, the inverse scattering procedure reveals that the radiation losses are zero for beams of planar symmetry, propagating in a Kerr nonlinearity. The equivalent (nonlinear induced) linear waveguide thus has periodic modes which are lossless — a fact apparently unknown within linear theory.

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[14] In general, the modes of periodic (linear) waveguides have a complicated form but, when the periodicity is a perturbation, an analytical solution is available. It should, however, be emphasized that, even in this perturbation limit, the intensity profiles of the periodic modes can change dramatically with propagation. Indeed, when near-resonance conditions apply, the periodic modes have intensity profiles which correspond exactly to the beating of two modes a and b of the axially uniform waveguide defined by the averaged periodic waveguide, i.e., $|E(x,y,z)|^2 = |a\Psi_a(x,y)\exp(i\beta_a z) + b\Psi_b(x,y)\exp(i\beta_b z)|^2$. Here, however, b/a is not arbitrary but is restricted to two values determined by the perturbation as discussed below. Consider, as a particular example, the periodic modes of a sinusoidally perturbed (linear) waveguide whose refractive index profile has the form $n^2(x,y,z) = \overline{n}^2(x,y)+\delta n^2(x,y)\cos(\Omega z+\phi)$, where all quantities are real, $\overline{n}^2$ represents the averaged or unperturbed waveguide, $\delta n^2 \ll \overline{n}^2$ and Ω denotes the spatial frequency of the perturbation. This periodic waveguide has two periodic modes, each of which is composed of two modes a and b of the (axially uniform) averaged waveguide, i.e., $E_\pm(x,y,z) = \{a\Psi_a(x,y)\exp(i\Omega z/2) + b\Psi_b(x,y)\exp(-i\Omega z/2)\} \times \exp(i\beta_\pm z)$, where $\beta_\pm = k\overline{n}_0 \pm C\sqrt{1+M^2}$, $\overline{n}_0 = \overline{n}(0,0)$ and $b/a = \left\{-M \pm \sqrt{1+M^2}\right\}\exp(-i\phi)$ when the fields Ψ_a, Ψ_b are normalized as $\int_{A_\infty} \Psi_{a,b}^2\, dA = 1$. The mismatch parameter $M = (\beta_a - \beta_b - \Omega)/(2C)$, with the magnitude of the perturbation manifested through the coupling coefficient $C = \{k/(4\overline{n}_0)\}\left|\int_{A_\infty} \delta n^2 \Psi_a \Psi_b\, dA\right|$. Now, when $\delta n^2 \to 0$, it follows that $C \to 0$. It is then seen from the expressions for b/a and M that near-resonance conditions $\Omega \simeq \beta_a - \beta_b$ must hold if $|b/a|$ is to be significant but finite. Otherwise, the periodic modes are the individual modes of the unperturbed axially uniform waveguide. In fact, this nearly degenerate perturbation theory becomes inaccurate when $|a|$ differs significantly from $|b|$, but good qualitative agreement exists for arbitrary b/a. This fully describes the periodic modes of a sinusoidally perturbed linear waveguide. For self-consistency, a periodic soliton must be a periodic mode of the (linear) waveguide it induces. Provided $n^2(|E|^2)$ is approximately sinusoidal, the soliton-induced waveguide will be of the form discussed above. Now, b/a sets the mismatch parameter M, the phase, ϕ, and, through the nonlinearity, the perturbation amplitude δn^2. We can then determine (via M) the spatial frequency Ω of the periodic perturbation that is required for the soliton intensity to represent a self-consistent solution. Any deviation from periodicity will lead to coupling between the a and b modes that comprise the periodic mode (Ref. [12], Chap. 27).

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[19] This is not exactly true within Maxwell's equations but it is a good approximation. Recall that optical solitons are described by Maxwell's equations which, in turn, reduce to the Helmholtz equation $(\nabla^2 + k^2n^2)\Psi = 0$ for TE waves. This equation is not integrable and, thus, it is not believed [20,21] to exhibit higher-order solitons — consistent with our above conclusions derived from linear physics. However, the Helmholtz equation can be approximated by the nonlinear Schrödinger equation for one-dimensional waves in a Kerr nonlinearity provided $\delta n \ll 1$ and forward-to-backward wave coupling is negligible ($\Omega \ll 2kn$). Both of these conditions hold for the periodic perturbations considered here. In the present context, δn represents the difference between the maximum-induced refractive index and the background refractive index. The neglected forward-to-backward coupling has a power of order $(\delta n)^2$ and is independent of the propagation distance. In other words, radiation is expected but on a much longer scale than that observed from periodic solitons of the threshold medium.

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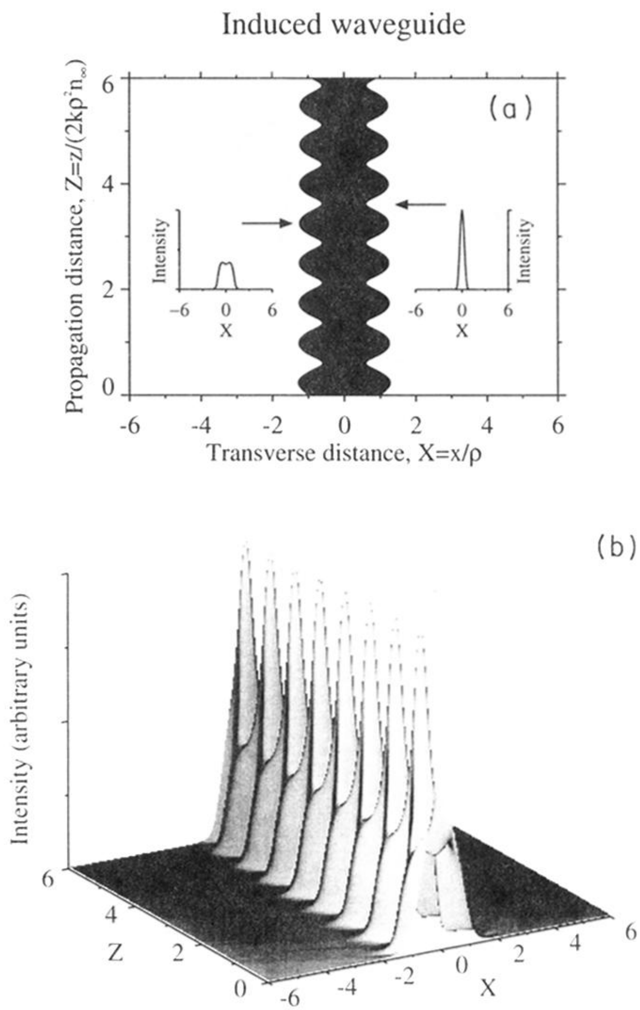

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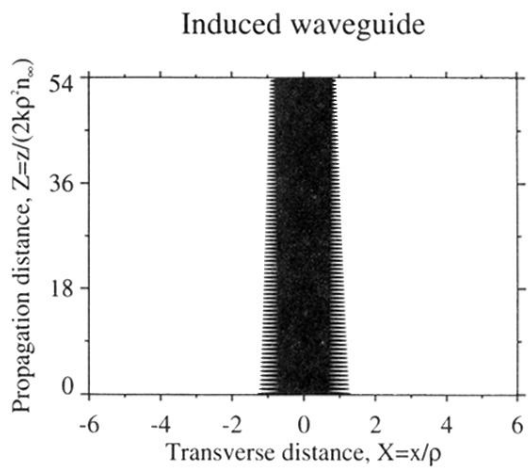

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